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The freeness and minimal free resolutions of modules of differential operators of a generic hyperplane arrangement

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Let $\mathcal{A}$ be a generic hyperplane arrangement composed of r hyperplanes in an n-dimensional vector space, and S the polynomial ring in n variables. We consider the S-submodule $D^{(m)}(\mathcal{A})$ of the nth Weyl algebra of homogeneous differential operators of order m preserving the defining ideal of $\mathcal{A}$.
We prove that if $n \geqslant 3$, $r > n$, $m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free (Holm's conjecture). Combining this with some results by Holm, we see that $D^{(m)}(\mathcal{A})$ is free unless $n \geqslant 3$, $r > n$, $m < r - n + 1$. In the remaining case, we construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ by generalizing Yuzvinsky's construction for $m = 1$. In addition, we construct a minimal free resolution of the transpose of the m-jet module, which generalizes a result by Rose and Terao for $m = 1$.

1. Introduction

In the study of a hyperplane arrangement, its derivation module plays a central character; in particular, its freeness over the polynomial ring attracts a great interest (see, e.g., Orlik and Terao [6]). Generalizing the study of the derivation module for a hyperplane arrangement to that of the modules of differential operators of higher order was initiated by Holm [4,5]. In particular, he studied the case of generic hyperplane arrangements in detail.

Let K denote a field of characteristic zero, and $\mathcal{A}$ a generic hyperplane arrangement in K^n composed of r hyperplanes. Let S be the polynomial ring $K[x_1, \dots, x_n]$, and $D^{(m)}(\mathcal{A})$ the S-module of homogeneous differential operators of order m of the hyperplane arrangement $\mathcal{A}$.

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Among others, in [5], Holm gave a finite generating set of the S-module $D^{(m)}(\mathcal{A})$. As to the freeness of $D^{(m)}(\mathcal{A})$, Holm [4] (cf. [9]) proved the following:

- If $n = 2$, then $D^{(m)}(\mathcal{A})$ is free for any m.
- If $n \geqslant 3$, $r > n$, $m < r - n + 1$, then $D^{(m)}(\mathcal{A})$ is not free.
- If $n \geqslant 3$, $r > n$, $m = r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free.

Holm also conjectured that if $n \geqslant 3$, $r > n$, $m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free.

Snellman [9] computed the Hilbert series of $D^{(m)}(\mathcal{A})$, which supported Holm's conjecture when $n \geqslant 3$, $r > n$, $m > r - n + 1$, and he conjectured the Poicaré–Betti series of $D^{(m)}(\mathcal{A})$ when $n \geqslant 3$, $r > n$, $m < r - n + 1$.

In the derivation module case, when $n \geqslant 3$, $r > n$, $m < r - n + 1$ with $m = 1$, Rose and Terao [7] and Yuzvinsky [11] independently gave a minimal free resolution of $D^{(1)}(\mathcal{A})$. In the course of the proof, Rose and Terao [7] gave minimal free resolutions of all modules of logarithmic differential forms with poles along $\mathcal{A}$. They also gave a minimal free resolution of S/J, where J is the Jacobian ideal of a polynomial defining $\mathcal{A}$. Yuzvinsky's construction [11] is more straightforward and combinatorial than [7].

In this paper, we prove Holm's conjecture, namely, we prove that if $n \geqslant 3$, $r > n$, $m > r - n + 1$, then $D^{(m)}(\mathcal{A})$ is free. Hence, for a generic hyperplane arrangement $\mathcal{A}$, $D^{(m)}(\mathcal{A})$ is free unless $n \geqslant 3$, $r > n$, $m < r - n + 1$. In the remaining case $n \geqslant 3$, $r > n$, $m < r - n + 1$, we construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ by generalizing [11] and a minimal free resolution of the transpose of the m-jet module generalizing that of S/J given by [7].

After we fix notation on differential operators for a hyperplane arrangement in Section 2, we recall the Saito–Holm criterion in Section 3. It was proved by Holm, and it is a criterion for a subset of $D^{(m)}(\mathcal{A})$ to form a basis, which generalizes the Saito criterion in the case of $m = 1$.

From Section 4 on, we assume that $r \geqslant n$ and the hyperplane arrangement $\mathcal{A}$ is generic. In Section 4, we recall the finite generating set of $D^{(m)}(\mathcal{A})$ given by Holm [5]. Then we recall the case $n = 2$ in Section 5 and the case $m = r - n + 1$ in Section 6 for completeness. In Section 7, we consider the case $m \geqslant r - n + 1$ and prove Holm's conjecture (Theorem 7.1).

From Section 8 on, we consider the case $m < r - n + 1$. In Section 8, we give a minimal generating set of $D^{(m)}(\mathcal{A})$ (Theorem 8.3). In Section 9, we generalize [11] to construct a minimal free resolution of $D^{(m)}(\mathcal{A})$ (Theorem 9.10). In Section 10, we generalize the minimal free resolution of S/J given in [7] (Theorem 10.7). In Section 11, we prove that the S-module considered in Section 10 is the transpose of the m-jet module $\Omega^{[1,m]}(S/SQ)$ (Theorem 11.2), where Q is a polynomial defining $\mathcal{A}$.

2. The modules of differential operators for a hyperplane arrangement

Throughout this paper, let K denote a field of characteristic zero, $\mathcal{A}$ a central hyperplane arrangement in K^n composed of r hyperplanes, and S the polynomial ring $K[x_1, \dots, x_n]$. We assume that $n \geqslant 2$.

For a hyperplane $H \in \mathcal{A}$, we fix a linear form $p_H \in S$ defining H. Set

$$Q := Q_{\mathcal{A}} := \prod_{H \in \mathcal{A}} p_H. \tag{2.1}$$

Let $D(S) = S\langle \partial_1, \dots, \partial_n \rangle$ denote the nth Weyl algebra, where $\partial_j = \frac{\partial}{\partial x_j}$. For a nonzero differential operator $P = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} f_{\boldsymbol{\alpha}}(x) \partial^{\boldsymbol{\alpha}} \in D(S)$, the maximum of $|\boldsymbol{\alpha}|$ with $f_{\boldsymbol{\alpha}} \neq 0$ is called the *order* of P, where

$$\partial^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \qquad |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$$

for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. If P has no nonzero $f_{\boldsymbol{\alpha}}$ with $|\boldsymbol{\alpha}| \neq m$, it is said to be *homogeneous of order* m. We denote by $D^{(m)}(S)$ the S-submodule of $D(S)$ of differential operators homogeneous of order m.

We denote by $*$ the action of $D(S)$ on S. For an ideal I of S,

$$D(I) := \{\theta \in D(S) \mid \theta * I \subseteq I\} \tag{2.2}$$

is called the *idealizer* of I.

We set

$$D(\mathcal{A}) := D(\langle Q \rangle). \tag{2.3}$$

Holm [5, Theorem 2.4] proved

$$D(\mathcal{A}) = \bigcap_{H \in \mathcal{A}} D(\langle p_H \rangle). \tag{2.4}$$

We denote by $D^{(m)}(\mathcal{A})$ the S-submodule of $D(\mathcal{A})$ of differential operators homogeneous of order m. Then Holm [5, Proposition 4.3] proved

$$D(\mathcal{A}) = \bigoplus_{m=0}^{\infty} D^{(m)}(\mathcal{A}).$$

A differential operator homogeneous of order 1 is nothing but a derivation. Hence $D^{(1)}(\mathcal{A})$ is the module of logarithmic derivations along $\mathcal{A}$.

The polynomial ring $S = \bigoplus_{p=0}^{\infty} S_p$ is a graded algebra, where S_p is the K-vector subspace spanned by the monomials of degree p. The nth Weyl algebra $D(S)$ is a graded S-module with $\deg(x^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}}) = |\boldsymbol{\alpha}| - |\boldsymbol{\beta}|$. Each $D^{(m)}(\mathcal{A})$ is a graded S-submodule of $D(S)$. An element $P = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} f_{\boldsymbol{\alpha}}(x) \partial^{\boldsymbol{\alpha}} \in D^{(m)}(\mathcal{A})$ is said to be *homogeneous of polynomial degree* p, and denoted by $\operatorname{pdeg} P = p$, if $f_{\boldsymbol{\alpha}} \in S_p$ for all $\boldsymbol{\alpha}$ with nonzero $f_{\boldsymbol{\alpha}}$.

3. Saito–Holm criterion

To prove that $D^{(1)}(\mathcal{A})$ is a free S-module, the Saito criterion ([8, Theorem 1.8(ii)], see also [6, Theorem 4.19]) is very useful. Holm [4] generalized the Saito criterion to the one for $D^{(m)}(\mathcal{A})$. In this section, we briefly review Holm's generalization.

Set

$$s_m := \binom{n+m-1}{m}, \qquad t_m := \binom{n+m-2}{m-1}.$$

Let

$$\{x^{\boldsymbol{\alpha}^{(1)}}, x^{\boldsymbol{\alpha}^{(2)}}, \ldots, x^{\boldsymbol{\alpha}^{(s_m)}}\}$$

be the set of monomials of degree m. For operators $\theta_1, \ldots, \theta_{s_m}$, define an $s_m \times s_m$ coefficient matrix $M_m(\theta_1, \ldots, \theta_{s_m})$ by

$$M_m(\theta_1, \ldots, \theta_{s_m}) := \begin{bmatrix} \theta_1 * \frac{x^{\boldsymbol{\alpha}^{(1)}}}{\boldsymbol{\alpha}^{(1)}!} & \cdots & \theta_{s_m} * \frac{x^{\boldsymbol{\alpha}^{(1)}}}{\boldsymbol{\alpha}^{(1)}!} \\ \vdots & \ddots & \vdots \\ \theta_1 * \frac{x^{\boldsymbol{\alpha}^{(s_m)}}}{\boldsymbol{\alpha}^{(s_m)}!} & \cdots & \theta_{s_m} * \frac{x^{\boldsymbol{\alpha}^{(s_m)}}}{\boldsymbol{\alpha}^{(s_m)}!} \end{bmatrix},$$

where $\boldsymbol{\alpha}! = (\alpha_1!)(\alpha_2!)\cdots(\alpha_n!)$ for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

The proofs of the following two propositions go similarly to those of [6, Proposition 4.12] and [6, Proposition 4.18].

Proposition 3.1. *(See III, Proposition 5.2 in [4] (cf. Proposition 4.12 in [6]).) If $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, then*

$$\det M_m(\theta_1, \dots, \theta_{s_m}) \in \langle Q^{t_m} \rangle.$$

Proposition 3.2. *(See III, Proposition 5.7 in [4] (cf. Proposition 4.18 in [6]).) Suppose that $D^{(m)}(\mathcal{A})$ is a free S-module. Then the rank of $D^{(m)}(\mathcal{A})$ is s_m.*

The following is a generalization of the Saito criterion. This was proved by Holm [4, III, Theorem 5.8].

Theorem 3.3 *(Saito–Holm criterion). Given $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$, the following two conditions are equivalent:*

(1) $\det M_m(\theta_1, \dots, \theta_{s_m}) = cQ_{\mathcal{A}}^{t_m}$ *for some* $c \in K^\times$,
(2) $\theta_1, \dots, \theta_{s_m}$ *form a basis for* $D^{(m)}(\mathcal{A})$ *over* S.

The following is an easy consequence of Theorem 3.3.

Theorem 3.4. *(See III, Theorem 5.9 in [4] (cf. Theorem 4.23 in [6]).) Let $\theta_1, \dots, \theta_{s_m} \in D^{(m)}(\mathcal{A})$ be linearly independent over S. Then $\theta_1, \dots, \theta_{s_m}$ form a basis for $D^{(m)}(\mathcal{A})$ over S if and only if*

$$\sum_{j=1}^{s_m} \operatorname{pdeg} \theta_j = rt_m.$$

Suppose that $D^{(m)}(\mathcal{A})$ is free over S. We denote by $\exp D^{(m)}(\mathcal{A})$ the multi-set of polynomial degrees of a basis for $D^{(m)}(\mathcal{A})$. The expression

$$\exp D^{(m)}(\mathcal{A}) = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \dots\}$$

means that $\exp D^{(m)}(\mathcal{A})$ has e_i i's $(i = 0, 1, 2, \dots)$.

Proposition 3.5. *(Cf. Proposition 4.26 in [6].) Assume that $D^{(m)}(\mathcal{A})$ is free over S, and suppose that*

$$\exp D^{(m)}(\mathcal{A}) = \{0^{e_0}, 1^{e_1}, 2^{e_2}, \dots\}.$$

Then

$$\sum_k e_k = s_m, \qquad \sum_k ke_k = rt_m.$$

Proof. Proposition 3.2 is the first statement, and Theorem 3.4 the second. □

4. Generic arrangements

In the rest of this paper, we assume that $r \geqslant n$ and $\mathcal{A}$ is generic. An arrangement $\mathcal{A}$ is said to be *generic*, if every n hyperplanes of $\mathcal{A}$ intersect only at the origin.

For a finite set $\mathcal{S}$, let $\mathcal{S}^{(k)} \subseteq 2^{\mathcal{S}}$ denote the set of $\mathcal{T} \subseteq \mathcal{S}$ with $\sharp\mathcal{T} = k$.

Given $\mathcal{H} \in \mathcal{A}^{(n-1)}$, the vector space

$$\left\{ \delta \in \sum_{i=1}^{n} K\partial_i \;\middle|\; \delta * p_H = 0 \text{ for all } H \in \mathcal{H} \right\}$$

is one-dimensional; fix a nonzero element $\delta_{\mathcal{H}}$ of this space. Note that

$$\delta_{\mathcal{H}} * p_H = 0 \quad \Leftrightarrow \quad H \in \mathcal{H}, \tag{4.1}$$

since $\mathcal{A}$ is generic.

For $\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$, which are not necessarily distinct, put

$$P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}} := \prod_{H \notin \bigcap_{i=1}^{m} \mathcal{H}_i} p_H. \tag{4.2}$$

Then $P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}}\delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} \in D^{(m)}(\mathcal{A})$ by (2.4). In particular, for $\mathcal{H} \in \mathcal{A}^{(n-1)}$,

$$P_{\mathcal{H}}\delta_{\mathcal{H}}^{m} \in D^{(m)}(\mathcal{A}),$$

where $P_{\mathcal{H}} := P_{\{\mathcal{H}\}}$. Note that

$$\deg P_{\mathcal{H}} = r - n + 1. \tag{4.3}$$

The operator

$$\epsilon_m := \sum_{|\boldsymbol{\alpha}|=m} \frac{m!}{\boldsymbol{\alpha}!} x^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \tag{4.4}$$

is called *the Euler operator of order m*. Then ϵ_1 is the Euler derivation, and $\epsilon_m = \epsilon_1(\epsilon_1 - 1)\cdots(\epsilon_1 - m + 1)$ [5, Lemma 4.9].

Holm gave a finite set of generators of $D^{(m)}(\mathcal{A})$ as an S-module:

Theorem 4.1. *(See Theorem 4.22 in [5].)*

$$D^{(m)}(\mathcal{A}) = \sum_{\mathcal{H}_1,\dots,\mathcal{H}_m \in \mathcal{A}^{(n-1)}} S P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}}\delta_{\mathcal{H}_1} \cdots \delta_{\mathcal{H}_m} + S\epsilon_m.$$

The following lemma will be used in Sections 7, 8, and 9.

Lemma 4.2. (1) *The set* $\{\delta_{\mathcal{H}}^{r-n+1} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ *is a* K*-basis of* $\sum_{|\boldsymbol{\alpha}|=r-n+1} K\partial^{\boldsymbol{\alpha}}$.

(2) *The set* $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ *is a* K*-basis of* $\sum_{|\boldsymbol{\alpha}|=r-n+1} Kx^{\boldsymbol{\alpha}} = S_{r-n+1}$.

Proof. The dimensions of $\sum_{|\boldsymbol{\alpha}|=r-n+1} K\partial^{\boldsymbol{\alpha}}$ and S_{r-n+1} are equal to

$$s_{r-n+1} = \binom{r}{r-n+1} = \binom{r}{n-1} = \sharp\mathcal{A}^{(n-1)}.$$

Let $\mathcal{H}, \mathcal{H}' \in \mathcal{A}^{(n-1)}$. Then

$$\delta_{\mathcal{H}}^{r-n+1} * P_{\mathcal{H}'} = \delta_{\mathcal{H}}^{r-n+1} * \prod_{H \notin \mathcal{H}'} p_H = \begin{cases} (r-n+1)! \prod_{H \notin \mathcal{H}} (\delta_{\mathcal{H}} * p_H) & \text{if } \mathcal{H}' = \mathcal{H}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

The assertions follow, since $\delta_{\mathcal{H}} * p_H = 0$ if and only if $H \in \mathcal{H}$. $\square$

5. The case $n = 2$

In this section, we consider central arrangements with $r \geqslant 2$ in K^2, which are always generic. Note that $s_m = m + 1$, and $t_m = m$.

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_r\}$. Put $p_i := p_{H_i}$, $P_i := P_{\{H_i\}}$, and $\delta_i := \delta_{\{H_i\}}$ for $i = 1, 2, \ldots, r$.

We may assume that there exist distinct $a_2, \ldots, a_r \in K$ such that

$$p_1 = x_1, \qquad p_i = x_2 - a_i x_1 \quad (i = 2, \ldots, r).$$

Then

$$\delta_1 = \partial_2, \qquad \delta_i = \partial_1 + a_i \partial_2 \quad (i = 2, \ldots, r),$$

and

$$P_i = Q/p_i \quad (i = 1, \ldots, r).$$

Proposition 5.1. *(See III, Proposition 6.7 in [4], Proposition 4.14 in [9].) The S-module $D^{(m)}(\mathcal{A})$ is free with the following basis:*

(1) $\{\epsilon_m, P_1\delta_1^m, \ldots, P_m\delta_m^m\}$ *if* $m \leqslant r - 2$.
(2) $\{P_1\delta_1^m, \ldots, P_r\delta_r^m\}$ *if* $m = r - 1$.
(3) $\{P_1\delta_1^m, \ldots, P_r\delta_r^m, Q\eta_{r+1}, \ldots, Q\eta_{m+1}\}$ *if* $m \geqslant r$, *where* $\{\delta_1^m, \ldots, \delta_r^m, \eta_{r+1}, \ldots, \eta_{m+1}\}$ *is a K-basis of* $\sum_{i=0}^m K\partial_1^i\partial_2^{m-i}$.

Corollary 5.2.

$$\exp D^{(m)}(\mathcal{A}) = \begin{cases} \{m^1, (r-1)^m\} & (1 \leqslant m \leqslant r-2), \\ \{(r-1)^{m+1}\} & (m = r-1), \\ \{(r-1)^r, r^{m-r+1}\} & (m \geqslant r). \end{cases}$$

6. The case $m = r - n + 1$

In this section, we consider the case $m = r - n + 1$. In this case,

$$s_m = \binom{n+m-1}{m} = \binom{r}{m} = \binom{r}{n-1}. \tag{6.1}$$

Note also that $\deg P_{\mathcal{H}} = r - n + 1 = m$ (4.3).

In Sections 7, 8, and 9, we use Lemma 4.2 in the case $m = r - n + 1$. Lemma 4.2 reads as follows in this case:

Lemma 6.1. (1) *The set $\{\delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K-basis of $\sum_{|\boldsymbol{\alpha}|=m} K\partial^{\boldsymbol{\alpha}}$.*

(2) *The set $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$ is a K-basis of $\sum_{|\boldsymbol{\alpha}|=m} Kx^{\boldsymbol{\alpha}} = S_m$.*

Proposition 6.2. *(See III, Proposition 6.8 in [4].) The S-module $D^{(m)}(\mathcal{A})$ is free with a basis $\{P_{\mathcal{H}}\delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}\}$.*

Corollary 6.3. *If $m = r - n + 1$, then*

$$\exp D^{(m)}(\mathcal{A}) = \left\{m^{\binom{r}{m}}\right\}.$$

7. The case $m \geqslant r - n + 1$

In this section, we assume that $m \geqslant r - n + 1$, and we prove Holm's conjecture by giving a basis of $D^{(m)}(\mathcal{A})$.

Set

$$\tilde{r} := n + m - 1,$$

and add $\tilde{r} - r$ hyperplanes to $\mathcal{A} = \{H_1, \ldots, H_r\}$ so that

$$\widetilde{\mathcal{A}} := \mathcal{A} \cup \{H_{r+1}, \ldots, H_{\tilde{r}}\} \tag{7.1}$$

is still generic.

For $\mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}$, define a homogeneous polynomial $P'_{\mathcal{H}} \in S$ by

$$P'_{\mathcal{H}} := \prod_{H \notin \mathcal{H};\, H \in \mathcal{A}} p_H. \tag{7.2}$$

Theorem 7.1. *The S-module $D^{(m)}(\mathcal{A})$ is free with a basis $\{P'_{\mathcal{H}} \delta^m_{\mathcal{H}} \mid \mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}\}$.*

Proof. By [5, Theorem 2.4], $P'_{\mathcal{H}} \delta^m_{\mathcal{H}} \in D^{(m)}(\mathcal{A})$ for each $\mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}$.

By Lemma 6.1(1), $\{P'_{\mathcal{H}} \delta^m_{\mathcal{H}} \mid \mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}\}$ is linearly independent over S. Since

$$\deg P'_{\mathcal{H}} = \sharp\{H \in \mathcal{A} \mid H \notin \mathcal{H}\},$$

the number of $\mathcal{H} \in \widetilde{\mathcal{A}}^{(n-1)}$ with $\deg P'_{\mathcal{H}} = j$, is

$$\binom{r}{j}\binom{\tilde{r} - r}{n - 1 - (r - j)} = \binom{r}{j}\binom{m + n - r - 1}{n - r + j - 1}.$$

Then

$$\begin{aligned}
\sum_j j\binom{r}{j}\binom{m + n - r - 1}{n - r + j - 1} &= r\sum_j \binom{r-1}{j-1}\binom{m + n - r - 1}{n - r + j - 1} \\
&= r\sum_j \binom{r-1}{j-1}\binom{m + n - r - 1}{m - j} \\
&= r\binom{m + n - 2}{m - 1} = rt_m.
\end{aligned}$$

Hence we have the assertion by Theorem 3.4. $\square$

Corollary 7.2.

$$\exp D^{(m)}(\mathcal{A}) = \left\{ j^{\binom{r}{j}\binom{m+n-r-1}{m-j}} \,\middle|\, r - n + 1 \leqslant j \leqslant \min\{r, m\} \right\}.$$

8. The case $m < r - n + 1$

Throughout this section, we assume that $m < r - n + 1$.

Recall that $D^{(m)}(\mathcal{A})$ is generated by

$$\left\{P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}}\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} \;\middle|\; \mathcal{H}_1,\dots,\mathcal{H}_m \in \mathcal{A}^{(n-1)}\right\} \cup \{\epsilon_m\} \tag{8.1}$$

over S (Theorem 4.1). In this section, we choose a minimal system of generators from (8.1) (Theorem 8.3), which implies that $D^{(m)}(\mathcal{A})$ is not free (Remark 8.5).

Note that

$$\sharp\mathcal{A}^{(n-1)} = \binom{r}{n-1} > \binom{n+m-1}{n-1} = s_m.$$

Lemma 8.1. *For any $\mathcal{H}_1,\dots,\mathcal{H}_m \in \mathcal{A}^{(n-1)}$, the following hold:*

(1) $$P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}} \in \bigcap_{\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} SP_{\mathcal{H}}.$$

(2) $$\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} \in \sum_{\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} K\delta_{\mathcal{H}}^m.$$

Proof. (1) If $\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}$, then $P_{\mathcal{H}} = \prod_{H \notin \mathcal{H}} p_H$ divides $\prod_{H \notin \bigcap_{i=1}^m \mathcal{H}_i} p_H = P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}}$. Hence the assertion is clear.

(2) Let $\bar{r} := n+m-1$. Take a subarrangement $\mathcal{B} \supset \bigcap_{i=1}^m \mathcal{H}_i$ of $\mathcal{A}$ with $\bar{r}$ hyperplanes. By Lemma 6.1, there exist $c_{\mathcal{H}} \in K$ ($\mathcal{H} \in \mathcal{B}^{(n-1)}$) such that

$$\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} = \sum_{\mathcal{H}\in\mathcal{B}^{(n-1)}} c_{\mathcal{H}}\delta_{\mathcal{H}}^m. \tag{8.2}$$

It suffices to show that $c_{\mathcal{H}} = 0$ for all $\mathcal{H} \not\supset \bigcap_{i=1}^m \mathcal{H}_i$. Fix $\mathcal{H} \in \mathcal{B}^{(n-1)}$ with $\mathcal{H} \not\supset \bigcap_{i=1}^m \mathcal{H}_i$, and put $\overline{P}_{\mathcal{H}} = \prod_{H\in\mathcal{B}\setminus\mathcal{H}} p_H$. Then $\deg \overline{P}_{\mathcal{H}} = \bar{r} - (n-1) = m$. Since there exists $H_0 \in (\bigcap_{i=1}^m \mathcal{H}_i) \setminus \mathcal{H}$, we have $\delta_{\mathcal{H}_i} * p_{H_0} = 0$ for all $i = 1,\dots,m$, and hence $\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} * \overline{P}_{\mathcal{H}} = 0$. Recall from (4.5) that

$$\delta_{\mathcal{H}'}^m * \overline{P}_{\mathcal{H}} = \begin{cases} m!\prod_{H\in\mathcal{B}\setminus\mathcal{H}}(\delta_{\mathcal{H}'} * p_H) \neq 0 & \text{if } \mathcal{H}' = \mathcal{H}, \\ 0 & \text{otherwise.} \end{cases}$$

Let the operator (8.2) act on $\overline{P}_{\mathcal{H}}$. Since

$$0 = \delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} * \overline{P}_{\mathcal{H}} = \sum_{\mathcal{H}\in\mathcal{B}^{(n-1)}} c_{\mathcal{H}}\delta_{\mathcal{H}}^m * \overline{P}_{\mathcal{H}} = c_{\mathcal{H}} \cdot m! \prod_{H\in\mathcal{B}\setminus\mathcal{H}} (\delta_{\mathcal{H}} * p_H),$$

we have $c_{\mathcal{H}} = 0$. $\square$

Proposition 8.2. *If $m < r - n + 1$, then*

$$D^{(m)}(\mathcal{A}) = \sum_{\mathcal{H}\in\mathcal{A}^{(n-1)}} SP_{\mathcal{H}}\delta_{\mathcal{H}}^m + S\epsilon_m.$$

Proof. Let $\mathcal{H}_1, \dots, \mathcal{H}_m \in \mathcal{A}^{(n-1)}$. By Lemma 8.1,

$$P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}}\delta_{\mathcal{H}_1}\cdots\delta_{\mathcal{H}_m} \in P_{\{\mathcal{H}_1,\dots,\mathcal{H}_m\}} \cdot \sum_{\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} K\delta_{\mathcal{H}}^m$$
$$\subset \sum_{\bigcap_{i=1}^m \mathcal{H}_i \subset \mathcal{H} \in \mathcal{A}^{(n-1)}} SP_{\mathcal{H}}\delta_{\mathcal{H}}^m.$$

Hence we obtain the assertion from Theorem 4.1. □

The system of generators for $D^{(m)}(\mathcal{A})$ in Proposition 8.2 is still large. Next, *we fix an order of the hyperplanes in* $\mathcal{A}$: $\mathcal{A} = \{H_1, \dots, H_r\}$, and define an S-submodule $\Xi^{(m)}(\mathcal{A})$ of $D^{(m)}(\mathcal{A})$ by

$$\Xi^{(m)}(\mathcal{A}) := \{\theta \in D^{(m)}(\mathcal{A}) \mid \theta * (p_{H_1} \cdots p_{H_m}) = 0\}. \tag{8.3}$$

For $\mathcal{H} \in \mathcal{A}^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset$, we have $\delta_{\mathcal{H}} * p_{H_i} = 0$ for some $i \leqslant m$, and hence $P_{\mathcal{H}}\delta_{\mathcal{H}}^m \in \Xi^{(m)}(\mathcal{A})$. Furthermore we have the following.

Theorem 8.3. *If $m < r - n + 1$, then*

$$D^{(m)}(\mathcal{A}) = \Xi^{(m)}(\mathcal{A}) \oplus S\epsilon_m = \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1,\dots,H_m\} \neq \emptyset}} SP_{\mathcal{H}}\delta_{\mathcal{H}}^m \oplus S\epsilon_m.$$

Moreover, the set $\{P_{\mathcal{H}}\delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)},\ \mathcal{H} \cap \{H_1, \dots, H_m\} \neq \emptyset\}$ is a minimal system of generators for $\Xi^{(m)}(\mathcal{A})$ over S.

Proof. Let $\theta \in D^{(m)}(\mathcal{A})$. Then

$$\theta - \frac{1}{m!}\frac{\theta * (p_{H_1} \cdots p_{H_m})}{p_{H_1} \cdots p_{H_m}}\epsilon_m \in \Xi^{(m)}(\mathcal{A}),$$

since $\theta \in D^{(m)}(\mathcal{A}) \subset D^{(m)}(\langle p_{H_1} \cdots p_{H_m}\rangle)$ by (2.4). So we have $D^{(m)}(\mathcal{A}) = \Xi^{(m)}(\mathcal{A}) + S\epsilon_m$. Moreover, $\epsilon_m * (p_{H_1} \cdots p_{H_m}) = m! p_{H_1} \cdots p_{H_m} \neq 0$ implies that $\Xi^{(m)}(\mathcal{A}) \cap S\epsilon_m = 0$.

Next, we show the second equality. By Proposition 8.2, it suffices to show that

$$P_{\mathcal{H}_0}\delta_{\mathcal{H}_0}^m \in \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1,\dots,H_m\} \neq \emptyset}} SP_{\mathcal{H}}\delta_{\mathcal{H}}^m \oplus S\epsilon_m$$

for every $\mathcal{H}_0 \in \mathcal{A}^{(n-1)}$ with $\mathcal{H}_0 \cap \{H_1, \dots, H_m\} = \emptyset$. Put $\mathcal{B} := \mathcal{H}_0 \cup \{H_1, \dots, H_m\}$. By Proposition 6.2,

$$D^{(m)}(\mathcal{B}) = \bigoplus_{\mathcal{H} \in \mathcal{B}^{(n-1)}} S\overline{P}_{\mathcal{H}}\delta_{\mathcal{H}}^m,$$

where $\overline{P}_{\mathcal{H}} = \prod_{H \in \mathcal{B} \setminus \mathcal{H}} p_H$. Since $\epsilon_m \in D^{(m)}(\mathcal{B})$, there exist $c_{\mathcal{H}} \in S$ ($\mathcal{H} \in \mathcal{B}^{(n-1)}$) such that

$$\epsilon_m = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}}\overline{P}_{\mathcal{H}}\delta_{\mathcal{H}}^m. \tag{8.4}$$

(By looking at polynomial degrees, we see $c_{\mathcal{H}} \in K$.) Multiplying $q := \prod_{H \in \mathcal{A} \setminus \mathcal{B}} p_H$ from the left, we have

$$q\epsilon_m = \sum_{\mathcal{H} \in \mathcal{B}^{(n-1)}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m. \tag{8.5}$$

Let the operator (8.4) act on $\overline{P}_{\mathcal{H}_0}$. Since

$$0 \neq m! \overline{P}_{\mathcal{H}_0} = m! c_{\mathcal{H}_0} \overline{P}_{\mathcal{H}_0} \cdot \prod_{H \in \mathcal{B} \setminus \mathcal{H}_0} \delta_{\mathcal{H}_0} * p_H = m! c_{\mathcal{H}_0} \overline{P}_{\mathcal{H}_0} \cdot \prod_{i=1}^{m} (\delta_{\mathcal{H}_0} * p_{H_i}),$$

we have $c_{\mathcal{H}_0} \neq 0$. Hence, we have

$$P_{\mathcal{H}_0} \delta_{\mathcal{H}_0}^m = c_{\mathcal{H}_0}^{-1} \Bigg(q\epsilon_m - \sum_{\substack{\mathcal{H} \in \mathcal{B}^{(n-1)} \\ \mathcal{H} \neq \mathcal{H}_0}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m \Bigg) \in \sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset}} S P_{\mathcal{H}} \delta_{\mathcal{H}}^m \oplus S\epsilon_m.$$

Finally, we show the minimality. It suffices to show that the set $\{P_{\mathcal{H}} \delta_{\mathcal{H}}^m \mid \mathcal{H} \in \mathcal{A}^{(n-1)}, \mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset\}$ is linearly independent over K, since all $P_{\mathcal{H}} \delta_{\mathcal{H}}^m$ have the same polynomial degree. Suppose that

$$\sum_{\substack{\mathcal{H} \in \mathcal{A}^{(n-1)} \\ \mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset}} c_{\mathcal{H}} P_{\mathcal{H}} \delta_{\mathcal{H}}^m = 0 \quad (c_{\mathcal{H}} \in K). \tag{8.6}$$

Fix arbitrary hyperplanes $H_{i_1}, \ldots, H_{i_m} \in \mathcal{A}$, and put $q' := p_{H_{i_1}} \cdots p_{H_{i_m}}$ and $\mathcal{B}' := \mathcal{A} \setminus \{H_{i_1}, \ldots, H_{i_m}\}$. Let the operator (8.6) act on q'. Then we have

$$\sum_{\substack{\mathcal{H} \in \mathcal{B}'^{(n-1)} \\ \mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset}} c_{\mathcal{H}} P_{\mathcal{H}} \prod_{\nu=1}^{m} (\delta_{\mathcal{H}} * p_{H_{i_\nu}}) = 0.$$

By Lemma 4.2, the set $\{P_{\mathcal{H}} \mid \mathcal{H} \in \mathcal{B}'^{(n-1)}\}$ is linearly independent over K. Hence $c_{\mathcal{H}} = 0$ for $\mathcal{H} \in \mathcal{B}'^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset$. For $\mathcal{H} \in \mathcal{A}^{(n-1)}$ with $\mathcal{H} \cap \{H_1, \ldots, H_m\} \neq \emptyset$, we may take $H_{i_1}, \ldots, H_{i_m} \in \mathcal{A}$ so that $\mathcal{H} \in \mathcal{B}'^{(n-1)}$, since $r > m + n - 1$. Hence we have finished the proof. □

Corollary 8.4. *(Cf. Conjecture 6.8 in [9].) The S-module $\Xi^{(m)}(\mathcal{A})$ is minimally generated by $\binom{r}{n-1} - \binom{r-m}{n-1}$ operators of polynomial degree $r - n + 1$.*

Remark 8.5. We can show

$$\binom{r}{n-1} - \binom{r-m}{n-1} + 1 > \binom{n+m-1}{n-1},$$

supposing that $m < r - n + 1$. Then by Proposition 3.2 and Corollary 8.4 we see that, for $n \geqslant 3$ and $m < r - n + 1$, $D^{(m)}(\mathcal{A})$ is not free over S, which was proved by Holm [4, III, Proposition 6.8].

9. Generalization of Yuzvinsky's paper [11]

In this section, we assume $m \leqslant r-n+1$, and we construct a minimal free resolution of $\Xi^{(m)}(\mathcal{A})$ when $m < r-n+1$ and $n \geqslant 3$. We generalize the construction in [11] step by step, and basically we follow Yuzvinsky's notation.

Let $V := K^n$. Recall that, for $\mathcal{H} \in \mathcal{A}^{(n-1)}$, $\delta_{\mathcal{H}} \in (V^*)^* = V$ is a nonzero derivation with constant coefficients such that $\delta_{\mathcal{H}} * p_H = 0$ for all $H \in \mathcal{H}$. Under the identification $(V^*)^* = V$, $K\delta_{\mathcal{H}}$ corresponds to the linear subspace $[\mathcal{H}] := \bigcap_{H\in\mathcal{H}} H = \bigcap_{H\in\mathcal{H}}(p_H = 0)$ of V. Similarly, $\mathcal{H} \in \mathcal{A}^{(n-j)}$ corresponds to the linear subspace $[\mathcal{H}] = \bigcap_{H\in\mathcal{H}} H \in L_j$, where L_j is the set of elements of dimension j of the intersection lattice of $\mathcal{A}$.

For $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $1 \leqslant j \leqslant n$, set

$$\Delta_{\mathcal{H}} := \sum_{\mathcal{H}' \in (\mathcal{A}\setminus\mathcal{H})^{(j-1)}} K\delta^m_{\mathcal{H}\cup\mathcal{H}'}.$$

Note that

$$\Delta_{\mathcal{H}} = K\delta^m_{\mathcal{H}} \quad \text{for } \mathcal{H} \in \mathcal{A}^{(n-1)},$$

and

$$\Delta_{\emptyset} = \sum_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K\delta^m_{\mathcal{H}}.$$

Each $\Delta_{\mathcal{H}}$ is a subspace of $\Delta_{\emptyset}$.

Example 9.1. Let $m = 1$. Then

$$\Delta_{\mathcal{H}} = \bigl\{\delta \in (V^*)^* \bigm| \delta * p_H = 0 \text{ for all } H \in \mathcal{H}\bigr\}.$$

Hence, under the identification $(V^*)^* = V$, $\Delta_{\mathcal{H}}$ corresponds to $[\mathcal{H}] = \bigcap_{H\in\mathcal{H}} H = \bigcap_{H\in\mathcal{H}}(p_H = 0)$.

Lemma 9.2. *Let $1 \leqslant j \leqslant n$, and let $\mathcal{H} \in \mathcal{A}^{(n-j)}$.*

Take $\mathcal{A}' := \{H_1, H_2, \ldots, H_{\bar{r}}\} \subseteq \mathcal{A}$ with $\bar{r} = m+n-1$ so that $\mathcal{H} \subseteq \mathcal{A}'$.

Then $\{\delta^m_{\mathcal{H}\cup\mathcal{H}'} \mid \mathcal{H}' \in (\mathcal{A}'\setminus\mathcal{H})^{(j-1)}\}$ forms a basis of $\Delta_{\mathcal{H}}$, and $\dim \Delta_{\mathcal{H}} = \binom{\bar{r}-(n-j)}{j-1} = \binom{m+j-1}{j-1}$.

Proof. By Lemma 6.1,

$$\sum_{|\boldsymbol{\alpha}|=m} K\partial^{\boldsymbol{\alpha}} = \bigoplus_{\mathcal{H}'' \in (\mathcal{A}')^{(n-1)}} K\delta^m_{\mathcal{H}''}. \tag{9.1}$$

Hence $\delta^m_{\mathcal{H}\cup\mathcal{H}'}$ $(\mathcal{H}' \in (\mathcal{A}'\setminus\mathcal{H})^{(j-1)})$ are linearly independent.

Let $\mathcal{H}''' \in (\mathcal{A}\setminus\mathcal{H})^{(j-1)} \setminus (\mathcal{A}')^{(j-1)}$. Then

$$\delta^m_{\mathcal{H}\cup\mathcal{H}'''} = \sum_{\mathcal{H}'' \in (\mathcal{A}')^{(n-1)}} \frac{\delta^m_{\mathcal{H}\cup\mathcal{H}'''} * P'_{\mathcal{H}''}}{\delta^m_{\mathcal{H}''} * P'_{\mathcal{H}''}} \delta^m_{\mathcal{H}''}, \tag{9.2}$$

where

$$P'_{\mathcal{H}''} := \prod_{H \in \mathcal{A}'\setminus\mathcal{H}''} p_H. \tag{9.3}$$

For $\mathcal{H}'' \not\supseteq \mathcal{H}$, there exists $H \in \mathcal{H} \setminus \mathcal{H}''$. Then p_H divides $P'_{\mathcal{H}''}$, and hence $\delta^m_{\mathcal{H}\cup\mathcal{H}'''} * P'_{\mathcal{H}''} = 0$. Therefore

$$\delta^m_{\mathcal{H}\cup\mathcal{H}'''} = \sum_{\mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}} \frac{\delta^m_{\mathcal{H}\cup\mathcal{H}'''} * P'_{\mathcal{H}\cup\mathcal{H}'}}{\delta^m_{\mathcal{H}\cup\mathcal{H}'} * P'_{\mathcal{H}\cup\mathcal{H}'}} \delta^m_{\mathcal{H}\cup\mathcal{H}'}. \tag{9.4}$$

Hence $\{\delta^m_{\mathcal{H}\cup\mathcal{H}'} \mid \mathcal{H}' \in (\mathcal{A}' \setminus \mathcal{H})^{(j-1)}\}$ forms a basis of $\Delta_{\mathcal{H}}$, and $\dim \Delta_{\mathcal{H}} = \binom{\bar{r}-(n-j)}{j-1} = \binom{m+j-1}{j-1}$. $\square$

Let $\mathcal{A} = \{H_1, H_2, \ldots, H_r\}$. We write $H_i \prec H_j$ if $i < j$.

We define the complex $C_*(\mathcal{A}) = C_*$ as follows. For $j = 1, 2, \ldots, n$, set

$$C_{n-j} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} \Delta_{\mathcal{H}} \boldsymbol{e}_{\wedge\mathcal{H}},$$

where $\boldsymbol{e}_{\wedge\mathcal{H}}$ is just a symbol. In particular,

$$C_{n-1} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K\delta^m_{\mathcal{H}} \boldsymbol{e}_{\wedge\mathcal{H}},$$

and

$$C_0 := \Delta_\emptyset \boldsymbol{e}_{\wedge\emptyset}.$$

The differential $\partial_j : C_j \to C_{j-1}$ is defined by

$$C_j = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \boldsymbol{e}_{\wedge\mathcal{H}} \ni \xi \boldsymbol{e}_{\wedge\mathcal{H}} \mapsto \sum_{H \in \mathcal{H}} (-1)^{l_{\mathcal{H}}(H)} \xi \boldsymbol{e}_{\wedge(\mathcal{H}\setminus\{H\})} \in C_{j-1},$$

where

$$l_{\mathcal{H}}(H) := \sharp\{H' \in \mathcal{H} \mid H' \prec H\}.$$

Set

$$C_n := \operatorname{Ker} \partial_{n-1}.$$

Lemma 9.3. *(Cf. Lemma 1.1 in [11].) The sequence C_* is exact.*

Proof. As in [11, Lemma 1.1], we prove the assertion by induction.

Let $r = m + n - 1$. Then by Lemma 6.1

$$\Delta_{\mathcal{H}} = \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)}} K\delta^m_{\mathcal{H}\cup\mathcal{H}'} \quad \text{for } \mathcal{H} \in \mathcal{A}^{(n-j)}.$$

Hence

$$C_{n-j} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-j)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(j-1)}} K\delta^m_{\mathcal{H}\cup\mathcal{H}'} \boldsymbol{e}_{\wedge\mathcal{H}} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-1)}} K\delta^m_{\mathcal{H}} \otimes \left(\bigoplus_{\mathcal{H}' \in \mathcal{H}^{(n-j)}} K\boldsymbol{e}_{\wedge\mathcal{H}'} \right).$$

Thus, in this case, with $C_n = 0$,

$$C_* = \bigoplus_{\mathcal{H}\in\mathcal{A}^{(n-1)}} K\delta_{\mathcal{H}}^m \otimes \widetilde{S}(\mathcal{H}),$$

where $\widetilde{S}(\mathcal{H})$ is the augmented chain complex of the simplex with vertex set $\mathcal{H}$. Hence C_* is exact.

For $n = 2$, the sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\partial_1 \arrow[r] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r] \arrow[d, equal] & 0 \\ & & \bigoplus_{H\in\mathcal{A}} K\delta_H^m \arrow[r] & \sum_{H\in\mathcal{A}} K\delta_H^m & \end{tikzcd}$$

is clearly exact.

Suppose that $n > 2$ and $r > m + n - 1$. Consider the arrangements $\mathcal{A}\setminus\{H_r\}$ and $\mathcal{A}^{H_r}$. Since $r > m + n - 1$, we have $\Delta_{\mathcal{H}}(\mathcal{A}) = \Delta_{\mathcal{H}}(\mathcal{A}\setminus\{H_r\})$ for $\mathcal{H} \in (\mathcal{A}\setminus\{H_r\})^{(n-j)}$ by Lemma 9.2. Hence

$$0 \to C_*\big(\mathcal{A}\setminus\{H_r\}\big) \to C_*(\mathcal{A}) \to C_*\big(\mathcal{A}^{H_r}\big)(-1) \to 0$$

is exact. We thus have the assertion by induction. □

Let $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $j = 1, 2, \ldots, n$, and let $C_*^{[\mathcal{H}]} := C_*(\mathcal{A}^{[\mathcal{H}]})$. For $\mathcal{H}' \in (\mathcal{A}\setminus\mathcal{H})^{(j-t)}$, we have

$$\Delta_{\mathcal{H}'}\big(\mathcal{A}^{[\mathcal{H}]}\big) = \sum_{\mathcal{H}''\in(\mathcal{A}\setminus\mathcal{H}\cup\mathcal{H}')^{(t-1)}} K\big(\delta^{[\mathcal{H}]}_{\mathcal{H}'\cup\mathcal{H}''}\big)^m.$$

Since we may identify $\delta^{[\mathcal{H}]}_{\mathcal{H}'\cup\mathcal{H}''}$ with $\delta_{\mathcal{H}\cup\mathcal{H}'\cup\mathcal{H}''}$, we may identify $\Delta_{\mathcal{H}'}(\mathcal{A}^{[\mathcal{H}]})$ with $\Delta_{\mathcal{H}\cup\mathcal{H}'}$. Hence

$$C^{[\mathcal{H}]}_{j-t} = \bigoplus_{\mathcal{H}'\in(\mathcal{A}\setminus\mathcal{H})^{(j-t)}} \Delta_{\mathcal{H}\cup\mathcal{H}'}\boldsymbol{e}_{\wedge\mathcal{H}'}\boldsymbol{e}_{\mathcal{H}} \tag{9.5}$$

for $t = 1, 2, \ldots, j$, where $\boldsymbol{e}_{\mathcal{H}}$ is again a symbol.

We put

$$E_{[\mathcal{H}]} := C^{[\mathcal{H}]}_j := \operatorname{Ker}\big(\partial^{[\mathcal{H}]}_{j-1} : C^{[\mathcal{H}]}_{j-1} \to C^{[\mathcal{H}]}_{j-2}\big)$$

for $\mathcal{H} \in \mathcal{A}^{(n-j)}$ with $j \geqslant 2$, and

$$E_{[\mathcal{H}]} := K\delta^m_{\mathcal{H}}\boldsymbol{e}_{\mathcal{H}}$$

for $\mathcal{H} \in \mathcal{A}^{(n-1)}$. Then we put

$$E_j := \bigoplus_{\mathcal{H}\in\mathcal{A}^{(n-j)}} E_{[\mathcal{H}]}$$

for $j = 1, 2, \ldots, n$.

Remark 9.4. (Cf. Remark 1.2 in [11].) Let $1 \leqslant j \leqslant n$ and $\mathcal{H} \in \mathcal{A}^{(n-j)}$. Then

$$\dim E_{[\mathcal{H}]} = \binom{r-m-n+j-1}{j-1}.$$

Proof. By Lemma 9.3,

$$\dim E_{[\mathcal{H}]} = \dim C_j^{[\mathcal{H}]} = \sum_{l=1}^{j} (-1)^{l-1} \dim C_{j-l}^{[\mathcal{H}]}.$$

Then by Lemma 9.2

$$\dim C_{j-l}^{[\mathcal{H}]} = \binom{r-n+j}{j-l}\binom{m+l-1}{l-1}.$$

Hence

$$\begin{aligned}
\dim E_{[\mathcal{H}]} &= \sum_{l=1}^{j} (-1)^{l-1} \binom{m+l-1}{l-1}\binom{r-n+j}{j-l} \\
&= \sum_{l=1}^{j} (-1)^{l-1} \binom{m+l-2}{l-1}\binom{r-n+j-1}{j-l} = \cdots \\
&= \sum_{l=1}^{j} (-1)^{l-1} \binom{l-2}{l-1}\binom{r-m-n+j-1}{j-l} = \binom{r-m-n+j-1}{j-1}. \qquad \square
\end{aligned}$$

Let

$$\Delta_{ij} := \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}} \Delta_{\mathcal{H} \cup \mathcal{H}'} \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}}$$

for $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-1$ with $i+j \geqslant n$, and

$$\Delta_{in} := E_i = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} E_{[\mathcal{H}]}.$$

Then

$$\Delta_{ij} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_{i+j-n}^{[\mathcal{H}]}, \quad \text{and hence } \Delta_{i\bullet} = \bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_\bullet^{[\mathcal{H}]}\bigl(-(n-i)\bigr).$$

As differentials of $\Delta_{i\bullet}$, we take $(-1)^i$ times the differentials of $\bigoplus_{\mathcal{H} \in \mathcal{A}^{(n-i)}} C_\bullet^{[\mathcal{H}]}(-(n-i))$. We define a linear map $\phi(j)_i : \Delta_{ij} \to \Delta_{i-1\,j}$ for $0 \leqslant j \leqslant n-1$ by

$$\Delta_{ij} \ni \xi \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}} \mapsto \sum_{H \in \mathcal{H}'} (-1)^{l_{\mathcal{H}'}(H)} \xi \boldsymbol{e}_{\wedge (\mathcal{H}' \setminus \{H\})} \boldsymbol{e}_{\mathcal{H} \cup \{H\}} \in \Delta_{i-1\,j}$$

for $\mathcal{H} \in \mathcal{A}^{(n-i)}, \mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i+j-n)}$, and $\xi \in \Delta_{\mathcal{H} \cup \mathcal{H}'}$. We define $\psi_i : E_i \to E_{i-1}$ as the restriction of $\phi(n-1)_i$.

Then we have the double complex $\Delta_{\bullet\bullet}$:

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_n & \longrightarrow & \Delta_{n,n-1} & \longrightarrow & \Delta_{n,n-2} & \longrightarrow & \cdots & \longrightarrow & \Delta_{n,1} & \longrightarrow & \Delta_{n,0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_n} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & E_{n-1} & \longrightarrow & \Delta_{n-1,n-1} & \longrightarrow & \Delta_{n-1,n-2} & \longrightarrow & \cdots & \longrightarrow & \Delta_{n-1,1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{n-1}} & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & & & 0 & & \\
 & & \downarrow{\scriptstyle \psi_3} & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & E_2 & \longrightarrow & \Delta_{2,n-1} & \longrightarrow & \Delta_{2,n-2} & \longrightarrow & 0 & & & & \\
 & & \downarrow{\scriptstyle \psi_2} & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & E_1 & \longrightarrow & \Delta_{1,n-1} & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & & & \\
 & & 0 & & 0. & & & & & & & &
\end{array}$$

We add

$$\psi_1 : E_1 = \bigoplus_{\mathcal{H}\in\mathcal{A}^{(n-1)}} K\delta^m_{\mathcal{H}}\boldsymbol{e}_{\mathcal{H}} \ni \delta^m_{\mathcal{H}}\boldsymbol{e}_{\mathcal{H}} \mapsto \delta^m_{\mathcal{H}} \in E_0 := \Delta_{\emptyset} = \sum_{\mathcal{H}\in\mathcal{A}^{(n-1)}} K\delta^m_{\mathcal{H}}.$$

Lemma 9.5. *(Cf. Lemma 1.3 in [11].) The sequence*

$$E_* : 0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to 0$$

is exact.

Proof. All rows of $\Delta_{\bullet\bullet}$ are exact by Lemma 9.3 and the argument in the paragraph just after the proof of Lemma 9.3.

For $1 \leqslant j < n$, since we have

$$\begin{aligned}
\Delta_{ij} &= \bigoplus_{\mathcal{H}\in\mathcal{A}^{(n-i)}} \bigoplus_{\mathcal{H}'\in(\mathcal{A}\setminus\mathcal{H})^{(i+j-n)}} \Delta_{\mathcal{H}\cup\mathcal{H}'}\boldsymbol{e}_{\wedge\mathcal{H}'}\boldsymbol{e}_{\mathcal{H}} \\
&= \bigoplus_{\mathcal{H}\in\mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \otimes_K \left(\bigoplus_{\mathcal{H}'\in\mathcal{H}^{(i+j-n)}} K\boldsymbol{e}_{\wedge\mathcal{H}'}\boldsymbol{e}_{\mathcal{H}\setminus\mathcal{H}'} \right),
\end{aligned}$$

the jth column $\Delta_{\bullet j}$ is the same as $\bigoplus_{\mathcal{H}\in\mathcal{A}^{(j)}} \Delta_{\mathcal{H}} \otimes_K \widetilde{S}(\mathcal{H})$, where $\widetilde{S}(\mathcal{H})$ is the augmented chain complex of the simplex with vertex set $\mathcal{H}$:

$$0 \to K\boldsymbol{e}_{\wedge\mathcal{H}} \to \bigoplus_{B\in\mathcal{H}^{(j-1)}} K\boldsymbol{e}_{\wedge B} \to \bigoplus_{B\in\mathcal{H}^{(j-2)}} K\boldsymbol{e}_{\wedge B} \to \cdots \to \bigoplus_{H\in\mathcal{H}} K\boldsymbol{e}_H \to K\boldsymbol{e}_{\emptyset} \to 0.$$

Thus the jth columns ($1 \leqslant j \leqslant n-1$) are exact. The 0th column has the unique nonzero term $\Delta_{\emptyset}\boldsymbol{e}_{\emptyset}(=E_0)$ at $i=n$. Hence by the spectral sequence argument we see that E_* is exact. $\square$

Let $\sigma \subseteq \{1,2,\ldots,r\}$ and $\sigma \neq \emptyset$. Put

$$\mathcal{L}_j[\sigma] := \bigl\{\mathcal{H} \in \mathcal{A}^{(n-j)} \bigm| \mathcal{H} \cap \{H_i \mid i \in \sigma\} \neq \emptyset\bigr\}.$$

For $1 \leqslant j \leqslant n$,

$$E_j[\sigma] := \bigoplus_{\mathcal{H}\in\mathcal{L}_j[\sigma]} E_{[\mathcal{H}]}.$$

Then

$$E_n[\sigma] = 0, \qquad E_1[\sigma] = \bigoplus_{\mathcal{H}\in\mathcal{L}_1[\sigma]} K\delta^m_{\mathcal{H}}\boldsymbol{e}_{\mathcal{H}}.$$

We put

$$E_0[\sigma] := \sum_{\mathcal{H}\in\mathcal{L}_1[\sigma]} K\delta^m_{\mathcal{H}}.$$

We also put

$$E_j[\emptyset] := 0$$

for all j. Then $\{(E_*[\sigma], \psi_*[\sigma])\}$ is a subcomplex of $\{(E_*, \psi_*)\}$.

Lemma 9.6. *(Cf. Lemma 1.4 in [11].) For every σ with $|\sigma| \leqslant n+m-1$, $E_*[\sigma]$ is exact.*

Proof. We prove the assertion by induction on $|\sigma|$. If $|\sigma| = 0$, then the assertion is trivial.

When $n=2$, we have

$$0 \to E_1[\sigma] = \bigoplus_{H\in\mathcal{L}_1[\sigma]} K\delta^m_H\boldsymbol{e}_H = \bigoplus_{H\in\sigma} K\delta^m_H\boldsymbol{e}_H \to E_0[\sigma] = \sum_{H\in\mathcal{L}_1[\sigma]} K\delta^m_H = \sum_{H\in\sigma} K\delta^m_H \to 0.$$

This is an isomorphism, since $|\sigma| \leqslant 2+m-1$ (see Lemma 6.1).

Now assume that $|\sigma| \geqslant 1$ and $n \geqslant 3$. Fix $j \in \sigma$ and put $\tau := \sigma \setminus \{j\}$. Then $E_*[\tau]$ and $E_*[\{j\}] = E_*(\mathcal{A}^{H_j})$ (by (9.5)) are subcomplexes of $E_*[\sigma]$, which are exact by the induction hypothesis and Lemma 9.5. Moreover there exists an exact sequence of complexes:

$$0 \to E_*[\tau] \cap E_*\bigl[\{j\}\bigr] \to E_*[\tau] \oplus E_*\bigl[\{j\}\bigr] \to E_*[\sigma] \to 0.$$

Since $E_*[\tau] \cap E_*[\{j\}] = E_*[\tau](\mathcal{A}^{H_j})$ and $|\tau| \leqslant (n-1)+m-1$, we are done. $\square$

Put

$$\sigma_0 := \{1,2,\ldots,m\},$$

and

$$\overline{E}_* := E_*[\sigma_0].$$

We use notation

$$\psi_j : \overline{E}_j \to \overline{E}_{j-1} \quad (j = 1, 2, \ldots, n-1).$$

Put

$$F_j := S \otimes \overline{E}_j \qquad (j = 0, 1, \ldots, n-1).$$

Note that F_i is a submodule of

$$S\Delta_{i,n-1}[\sigma_0] = \bigoplus_{\mathcal{H} \in \mathcal{L}_i[\sigma_0]} \bigoplus_{\mathcal{H}' \in (\mathcal{A} \setminus \mathcal{H})^{(i-1)}} S\delta^m_{\mathcal{H} \cup \mathcal{H}'} \boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}}.$$

For $i \geqslant 2$, the morphism $d_i : F_i \to F_{i-1}$ is defined by

$$\boldsymbol{e}_{\wedge \mathcal{H}'} \boldsymbol{e}_{\mathcal{H}} \mapsto \sum_{H' \in \mathcal{H}'} (-1)^{l_{\mathcal{H}'}(H')} p_{H'} \boldsymbol{e}_{\wedge (\mathcal{H}' \setminus \{H'\})} \boldsymbol{e}_{\mathcal{H} \cup \{H'\}}.$$

Note that

$$F_0 = S \otimes_K \sum_{\mathcal{H} \in \mathcal{L}_1[\sigma_0]} K\delta^m_{\mathcal{H}},$$

and

$$F_1 = \bigoplus_{\mathcal{H} \in \mathcal{L}_1[\sigma_0]} S\delta^m_{\mathcal{H}} \boldsymbol{e}_{\mathcal{H}}.$$

We define a morphism $d_1 : F_1 \to F_0$ by

$$\delta^m_{\mathcal{H}} \boldsymbol{e}_{\mathcal{H}} \mapsto P_{\mathcal{H}} \delta^m_{\mathcal{H}}.$$

Lemma 9.7. *The sequence*

$$0 \to F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0$$

is a complex.

Proof. By the definition of d_i, clearly $d_i \circ d_{i+1} = 0$ for $i \geqslant 2$. We prove $d_1 \circ d_2 = 0$.

Let $X = \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} \delta^m_{\mathcal{H} \cup \{H\}} \boldsymbol{e}_{\wedge H} \boldsymbol{e}_{\mathcal{H}} \in F_2$. Then

$$\sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} \delta^m_{\mathcal{H} \cup \{H\}} = 0 \quad \text{for all } \mathcal{H} \in \mathcal{L}_2[\sigma_0].$$

We have

$$\begin{aligned} d_1 \circ d_2(X) &= d_1\left(\sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} p_H \delta^m_{\mathcal{H} \cup \{H\}} \boldsymbol{e}_{\mathcal{H} \cup \{H\}} \right) \\ &= \sum_{\mathcal{H} \in \mathcal{L}_2[\sigma_0]} \sum_{H \notin \mathcal{H}} f_{\mathcal{H},H} p_H P_{\mathcal{H} \cup \{H\}} \delta^m_{\mathcal{H} \cup \{H\}} \end{aligned}$$

$$= \sum_{\mathcal{H}\in\mathcal{L}_2[\sigma_0]} \sum_{H\notin\mathcal{H}} f_{\mathcal{H},H} P_{\mathcal{H}} \delta^m_{\mathcal{H}\cup\{H\}} \quad \Bigl(\text{here } P_{\mathcal{H}} := \prod_{H\notin\mathcal{H}} p_H\Bigr)$$
$$= \sum_{\mathcal{H}\in\mathcal{L}_2[\sigma_0]} P_{\mathcal{H}} \sum_{H\notin\mathcal{H}} f_{\mathcal{H},H} \delta^m_{\mathcal{H}\cup\{H\}} = 0. \qquad \square$$

The following is Theorem 8.3.

Lemma 9.8. *(Cf. Lemma 2.1 in [11].) Assume that $m < r - n + 1$. Then the image of d_1 coincides with $\Xi^{(m)}(\mathcal{A})$.*

By Remark 9.4, we have the following.

Remark 9.9. (Cf. Remark 2.2 in [11].)

$$\operatorname{rank}_S(F_j) = \binom{r-m-n+j-1}{j-1}\left(\binom{r}{n-j} - \binom{r-m}{n-j}\right) =: w_j^{(m)}.$$

Under the above preparations, we can prove the following theorem. Since the proof is almost the same as that of [11, Theorem 2.3], we omit it.

Theorem 9.10. *(Cf. Theorem 2.3 in [11].) Assume that $n \geqslant 3$ and $m < r - n + 1$. Then the complex*

$$F_* : 0 \to F_{n-1} \xrightarrow{d_{n-1}} F_{n-2} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} \Xi^{(m)}(\mathcal{A}) \to 0$$

is a minimal free resolution of $\Xi^{(m)}(\mathcal{A})$. In particular, the projective dimensions of S-modules $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to $n-2$.

By Theorem 8.3, Remark 9.9, and the construction of the complex F_* in Theorem 9.10, we have the following corollary:

Corollary 9.11. *(Cf. Corollary 4.4.3 in [7].) Assume that $n \geqslant 3$ and $m < r - n + 1$. Then there exist exact sequences*

$$0 \to S(m+1-r)^{w_{n-1}^{(m)}} \to \cdots \to S(m+n-j-r)^{w_j^{(m)}} \to \cdots$$
$$\to S(m+n-2-r)^{w_2^{(m)}} \to S(m+n-1-r)^{w_1^{(m)}} \to \Xi^{(m)}(\mathcal{A}) \to 0,$$
$$0 \to S(m+1-r)^{w_{n-1}^{(m)}} \to \cdots \to S(m+n-j-r)^{w_j^{(m)}} \to \cdots$$
$$\to S(m+n-2-r)^{w_2^{(m)}} \to S(m+n-1-r)^{w_1^{(m)}} \bigoplus S \to D^{(m)}(\mathcal{A}) \to 0,$$

where $w_j^{(m)}$ were defined in Remark 9.9*, and all maps are homogeneous of degree* 0.

In particular, the Castelnuovo–Mumford regularities of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to $r - m - n + 1$.

Remark 9.12. If we use the polynomial degrees in $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ as the degrees of graded S-modules, then the degrees are shifted by m. Then the Castelnuovo–Mumford regularities of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ are equal to $r - n + 1$ as stated for $D^{(1)}(\mathcal{A})$ in [1, Section 5.2], and the Poicaré–Betti series of $\Xi^{(m)}(\mathcal{A})$ and $D^{(m)}(\mathcal{A})$ coincide with the ones conjectured by Snellman [9, Conjecture 6.8].

10. Minimal free resolution of $J_m(\mathcal{A})$

In this section, we generalize the minimal free resolution of S/J given in [7], where J is the Jacobian ideal of Q. We retain the assumptions $n \geqslant 3$ and $n+m-1<r$.

Let $J_m(\mathcal{A})$ denote the S-submodule of $S^{\binom{n+m-1}{m-1}} = \bigoplus_{|\boldsymbol{\beta}| \leqslant m-1} S\boldsymbol{e}_{\boldsymbol{\beta}}$ generated by all

$$\frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}} \bullet Q := \left(\frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} * Q \colon |\boldsymbol{\beta}| \leqslant m-1\right) = \sum_{|\boldsymbol{\beta}| \leqslant m-1} \frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} * Q\,\boldsymbol{e}_{\boldsymbol{\beta}} \tag{10.1}$$

with $1 \leqslant |\boldsymbol{\alpha}| \leqslant m$. Here we agree $\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} = 0$ for $\boldsymbol{\beta} \nleqslant \boldsymbol{\alpha}$.

Example 10.1. Let $m=1$. Then $J_1(\mathcal{A})$ is the S-submodule of S generated by $\partial_j * Q$ $(j=1,\dots,n)$, i.e., $J_1(\mathcal{A})$ is nothing but the Jacobian ideal J of Q.

Lemma 10.2. *For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$,*

$$\frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}} = (-1)^{|\boldsymbol{\beta}|}\frac{(\operatorname{ad} x)^{\boldsymbol{\beta}}}{\boldsymbol{\alpha}!}(\partial^{\boldsymbol{\alpha}}).$$

Here we denote by $\operatorname{ad} x_i$ the endomorphism of $D(S)$: $D(S) \ni P \mapsto \operatorname{ad} x_i(P) = [x_i, P] \in D(S)$. For $\boldsymbol{\beta} = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$, we set $(\operatorname{ad} x)^{\boldsymbol{\beta}} = (\operatorname{ad} x_1)^{\beta_1} \circ \cdots \circ (\operatorname{ad} x_n)^{\beta_n}$.

Proof. We prove the assertion by induction on $|\boldsymbol{\beta}|$. For all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$,

$$\begin{aligned}
\operatorname{ad} x_i\left((-1)^{|\boldsymbol{\beta}|}\frac{(\operatorname{ad} x)^{\boldsymbol{\beta}}}{\boldsymbol{\alpha}!}(\partial^{\boldsymbol{\alpha}})\right) &= \frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\operatorname{ad} x_i\big(\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\big) \\
&= -\frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta})!}(\alpha_i-\beta_i)\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}-\mathbf{1}_i} \\
&= -\frac{1}{(\boldsymbol{\alpha}-\boldsymbol{\beta}-\mathbf{1}_i)!}\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}-\mathbf{1}_i}. \qquad \square
\end{aligned}$$

By Lemma 10.2,

$$\frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}} \bullet Q = \left((-1)^{|\boldsymbol{\beta}|}(\operatorname{ad} x)^{\boldsymbol{\beta}}\left(\frac{1}{\boldsymbol{\alpha}!}\partial^{\boldsymbol{\alpha}}\right) * Q \;\middle|\; |\boldsymbol{\beta}| \leqslant m-1\right).$$

We define an S-module morphism

$$\delta_0 : F_0^{[1,m]} := D^{[1,m]}(S) := \bigoplus_{k=1}^{m} D^{(k)}(S) \to S^{\binom{n+m-1}{m-1}} = \bigoplus_{|\boldsymbol{\beta}| \leqslant m-1} S\boldsymbol{e}_{\boldsymbol{\beta}}$$

by

$$\delta_0(\theta) := \theta \bullet Q := \big((-1)^{|\boldsymbol{\beta}|}(\operatorname{ad} x)^{\boldsymbol{\beta}}(\theta) * Q \;\big|\; |\boldsymbol{\beta}| \leqslant m-1\big). \tag{10.2}$$

By definition,

$$\operatorname{Im}\delta_0 = J_m(\mathcal{A}). \tag{10.3}$$

Lemma 10.3. *Let $\theta \in D(S)$. Then*

$$\theta x^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta),$$

where $\binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} = \prod_{i=1}^{n} \binom{\beta_i}{\gamma_i}$.

Proof. We prove the assertion by induction on $|\boldsymbol{\beta}|$. We have

$$\begin{aligned}
\theta x_i x^{\boldsymbol{\beta}} &= -\bigl(\operatorname{ad} x_i(\theta)\bigr) x^{\boldsymbol{\beta}} + x_i \theta x^{\boldsymbol{\beta}} \\
&= -\sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}} \bigl(\operatorname{ad} x_i(\theta)\bigr) + x_i \sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) \\
&= \sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}+\mathbf{1}_i|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}+\mathbf{1}_i-\boldsymbol{\gamma}-\mathbf{1}_i} (\operatorname{ad} x)^{\boldsymbol{\gamma}+\mathbf{1}_i}(\theta) + \sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}+\mathbf{1}_i-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) \\
&= \sum_{\boldsymbol{\gamma}-\mathbf{1}_i \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}-\mathbf{1}_i} x^{\boldsymbol{\beta}+\mathbf{1}_i-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) + \sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}+\mathbf{1}_i-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta) \\
&= \sum_{\boldsymbol{\gamma} \leqslant \boldsymbol{\beta}+\mathbf{1}_i} (-1)^{|\boldsymbol{\gamma}|} \binom{\boldsymbol{\beta}+\mathbf{1}_i}{\boldsymbol{\gamma}} x^{\boldsymbol{\beta}+\mathbf{1}_i-\boldsymbol{\gamma}} (\operatorname{ad} x)^{\boldsymbol{\gamma}}(\theta). \qquad \square
\end{aligned}$$

Let $\bar{\delta}_0$ denote the composite of δ_0 with the canonical projections of $S\boldsymbol{e}_{\boldsymbol{\beta}}$ onto $(S/SQ)\boldsymbol{e}_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \neq \mathbf{0}$:

$$\bar{\delta}_0 : D^{[1,m]}(S) \xrightarrow{\delta_0} \bigoplus_{|\boldsymbol{\beta}| \leqslant m-1} S\boldsymbol{e}_{\boldsymbol{\beta}} \to S\boldsymbol{e}_{\mathbf{0}} \bigoplus \bigoplus_{0 \neq |\boldsymbol{\beta}| \leqslant m-1} (S/SQ)\boldsymbol{e}_{\boldsymbol{\beta}}. \tag{10.4}$$

Here note that $\bar{\delta}_0$ is a graded S-module homomorphism homogeneous of degree 0 if we put $\deg(\boldsymbol{e}_{\boldsymbol{\beta}}) = -r - |\boldsymbol{\beta}|$.

In the following two lemmas, we describe the cokernel and the kernel of $\bar{\delta}_0$.

Lemma 10.4.

$$\operatorname{Coker} \bar{\delta}_0 = S^{\binom{n+m-1}{m-1}} / \bigl(J_m(\mathcal{A}) + Q\, S^{\binom{n+m-1}{m-1}}\bigr).$$

Proof. By (10.3), we only need to show $Q\boldsymbol{e}_{\mathbf{0}} \in \operatorname{Im} \bar{\delta}_0$. We have $\epsilon_1 * Q = rQ$. Since $\epsilon_1 \in D(\mathcal{A})$, we see $\delta_0(\epsilon_1) \in \bigoplus_{|\boldsymbol{\beta}| \leqslant m-1} SQ\boldsymbol{e}_{\boldsymbol{\beta}}$ by the definition of δ_0 (10.2). Hence

$$Q\boldsymbol{e}_{\mathbf{0}} = \bar{\delta}_0\left(\frac{1}{r}\epsilon_1\right) \in \operatorname{Im} \bar{\delta}_0. \qquad \square$$

Lemma 10.5.

$$\operatorname{Ker} \bar{\delta}_0 = \bigoplus_{k=1}^{m} D^{(k)}(\mathcal{A})' =: D^{[1,m]}(\mathcal{A})',$$

where $D^{(k)}(\mathcal{A})' := \{\theta \in D^{(k)}(\mathcal{A}) \colon \theta * Q = 0\}$.

Proof. If $\theta \in D^{[1,m]}(\mathcal{A})'$, then $(\operatorname{ad} x)^{\boldsymbol{\beta}}(\theta) \in D(\mathcal{A})$ for all $\boldsymbol{\beta}$, and $\theta * Q = 0$. Hence $\theta \in \operatorname{Ker} \bar{\delta}_0$ by the definitions of $D(\mathcal{A})$ and $\bar{\delta}_0$.

Next we suppose that $\theta \in \operatorname{Ker} \bar{\delta}_0$. Then by Lemma 10.3,

$$\theta * x^{\boldsymbol{\beta}} Q \in \langle Q \rangle = QS \quad \text{for all } \boldsymbol{\beta} \text{ with } |\boldsymbol{\beta}| \leqslant m-1. \tag{10.5}$$

By [5, Proposition 2.3], we conclude that $\theta \in D^{[1,m]}(\mathcal{A})'$. □

Lemma 10.6. *Let $k \leqslant r$. As S-modules,*

$$\Xi^{(k)}(\mathcal{A}) \simeq D^{(k)}(\mathcal{A})'.$$

Proof. It is easy to see that

$$\gamma_k : \Xi^{(k)}(\mathcal{A}) \ni \theta \mapsto \theta - \frac{\theta * Q}{Q} \frac{\epsilon_k}{r(r-1)\cdots(r-k+1)} \in D^{(k)}(\mathcal{A})'$$

and

$$D^{(k)}(\mathcal{A})' \ni \theta \mapsto \theta - \frac{\theta * (p_1 \cdots p_k)}{p_1 \cdots p_k} \frac{\epsilon_k}{k!} \in \Xi^{(k)}(\mathcal{A})$$

are inverse to each other. □

For $1 \leqslant k \leqslant m$, let $F_*^{(k)}$ denote the minimal free resolution of $\Xi^{(k)}$ in Theorem 9.10.

We consider the following complex:

$$0 \to \widetilde{F}_{n-1} \xrightarrow{\tilde{\delta}_{n-1}} \cdots \xrightarrow{\tilde{\delta}_2} \widetilde{F}_1 \xrightarrow{\tilde{\delta}_1} \widetilde{F}_0 \xrightarrow{\tilde{\delta}_0} \widetilde{F}_{-1} \to \operatorname{Coker}(\tilde{\delta}_0) \to 0, \tag{10.6}$$

where

$$\widetilde{F}_{-1} = \bigoplus_{|\boldsymbol{\beta}| \leqslant m-1} S\boldsymbol{e}_{\boldsymbol{\beta}},$$

$$\widetilde{F}_0 = D^{[1,m]}(S) \bigoplus \bigoplus_{0 \neq |\boldsymbol{\beta}| \leqslant m-1} S\boldsymbol{e}_{\boldsymbol{\beta}},$$

$$\widetilde{F}_j = \bigoplus_{k=1}^{m} F_j^{(k)} \quad (j = 1, \ldots, n-1),$$

and

$$\tilde{\delta}_0\Bigl(\theta, \sum_{\boldsymbol{\beta} \neq \boldsymbol{0}} f_{\boldsymbol{\beta}} \boldsymbol{e}_{\boldsymbol{\beta}}\Bigr) = \delta_0(\theta) + \sum_{\boldsymbol{\beta} \neq \boldsymbol{0}} f_{\boldsymbol{\beta}} Q \boldsymbol{e}_{\boldsymbol{\beta}} = \theta * Q \boldsymbol{e}_{\boldsymbol{0}} + \sum_{\boldsymbol{\beta} \neq \boldsymbol{0}} \bigl((-1)^{|\boldsymbol{\beta}|} (\operatorname{ad} x)^{\boldsymbol{\beta}}(\theta) * Q + f_{\boldsymbol{\beta}} Q\bigr)\boldsymbol{e}_{\boldsymbol{\beta}},$$

$$\tilde{\delta}_1\bigl(\delta_{\mathcal{H}}^k \boldsymbol{e}_{\mathcal{H}}^{(k)}\bigr) = \Bigl(\gamma_k\bigl(P_{\mathcal{H}} \delta_{\mathcal{H}}^k\bigr), -\frac{1}{Q} \sum_{\boldsymbol{\beta} \neq \boldsymbol{0}} (-1)^{|\boldsymbol{\beta}|} (\operatorname{ad} x)^{\boldsymbol{\beta}} \bigl(\gamma_k\bigl(P_{\mathcal{H}} \delta_{\mathcal{H}}^k\bigr)\bigr) * Q \boldsymbol{e}_{\boldsymbol{\beta}}\Bigr),$$

$$\tilde{\delta}_j = \bigoplus_{k=1}^{m} d_j^{(k)} \quad (j \geqslant 2).$$

Recall that $D^{(k)}(S) = F_0^{(k)}$, and $d_1(\delta_{\mathcal{H}}^k \boldsymbol{e}_{\mathcal{H}}^{(k)}) = P_{\mathcal{H}} \delta_{\mathcal{H}}^k$ for $1 \leqslant k \leqslant m$.

Theorem 10.7. *(Cf. Theorem 4.5.3 in [7].) The complex* (10.6) *is a minimal free resolution of* $\mathrm{Coker}(\bar{\delta}_0) = S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + Q\,S^{\binom{n+m-1}{m-1}})$.

Proof. The complex (10.6) is exact by Theorem 8.3, Theorem 9.10, Lemma 10.4, Lemma 10.5, and Lemma 10.6. The operator $P_{\mathcal{H}}\delta_{\mathcal{H}}^k$ is of order k and homogeneous of polynomial degree $\deg(P_{\mathcal{H}}) = r-(n-1)$. Then each term of $\gamma_k(P_{\mathcal{H}}\delta_{\mathcal{H}}^k)$ is of order k and of polynomial degree greater than or equal to k. Hence each term of the operator $(\mathrm{ad}\,x)^{\boldsymbol{\beta}}(\gamma_k(P_{\mathcal{H}}\delta_{\mathcal{H}}^k))$ is of order $k-|\boldsymbol{\beta}|$ and of polynomial degree greater than or equal to k. Therefore each term of the polynomial

$$\frac{1}{Q}(-1)^{|\boldsymbol{\beta}|}(\mathrm{ad}\,x)^{\boldsymbol{\beta}}\bigl(\gamma_k\bigl(P_{\mathcal{H}}\delta_{\mathcal{H}}^k\bigr)\bigr) * Q$$

is of degree greater than or equal to

$$r-\bigl(k-|\boldsymbol{\beta}|\bigr)+k-r = |\boldsymbol{\beta}| > 0.$$

Thus the free resolution (10.6) of $\mathrm{Coker}(\tilde{\delta}_0)$ is minimal. Clearly by (10.4)

$$\mathrm{Coker}(\tilde{\delta}_0) = S^{\binom{n+m-1}{m-1}}/\bigl(J_m(\mathcal{A}) + Q\,S^{\binom{n+m-1}{m-1}}\bigr) = \mathrm{Coker}(\bar{\delta}_0). \qquad \square$$

The following corollary is clear from Theorem 10.7 and the Auslander–Buchsbaum formula.

Corollary 10.8. *(Cf. Corollary 4.5.5 [7].) The projective dimension of the S-module $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + Q\,S^{\binom{n+m-1}{m-1}})$ is n, and the depth is* 0.

In the complex (10.6), the degrees of elements of bases are as follows:

$$\begin{aligned}
&\deg(\boldsymbol{e}_{\boldsymbol{\beta}}) = -r-|\boldsymbol{\beta}| && \text{in } \tilde{F}_{-1},\\
&\deg\bigl(\partial^{\boldsymbol{\alpha}}\bigr) = -|\boldsymbol{\alpha}| && \text{in } \widetilde{F}_0,\\
&\deg(\boldsymbol{e}_{\boldsymbol{\beta}}) = -|\boldsymbol{\beta}| && \text{in } \tilde{F}_0,\\
&\deg\bigl(\delta_{\mathcal{H}}^k\boldsymbol{e}_{\mathcal{H}}\bigr) = -k+r-(n-1) = r-n-k+1 && \text{in } \widetilde{F}_1.
\end{aligned}$$

Hence we have the following corollary:

Corollary 10.9. *(Cf. Corollary 4.5.4 in [7].) Assume that $n \geqslant 3$ and $m < r-n+1$. Then there exists an exact sequence*

$$\begin{aligned}
0 &\to \bigoplus_{k=1}^{m} S(k+1-r)^{w_{n-1}^{(k)}} \to \cdots \to \bigoplus_{k=1}^{m} S(k+n-j-r)^{w_j^{(k)}} \to \cdots\\
&\to \bigoplus_{k=1}^{m} S(k+n-1-r)^{w_1^{(k)}} \to \bigoplus_{k=1}^{m} S(k)^{s_k} \bigoplus \bigoplus_{k=1}^{m-1} S(k)^{s_k}\\
&\to \bigoplus_{k=0}^{m-1} S(r+k)^{s_k} \to \mathrm{Coker}(\bar{\delta}_0) \to 0,
\end{aligned}$$

where $w_j^{(k)}$ were defined in Remark 9.9, *$s_k = \binom{n+k-1}{k}$, and all maps are homogeneous of degree* 0.

In particular, the Castelnuovo–Mumford regularity of $\mathrm{Coker}(\bar{\delta}_0)$ *is equal to $r-n-2$.*

Remark 10.10. In Corollary 10.9, to make the degrees of all the minimal generators of $\mathrm{Coker}(\bar{\delta}_0)$ nonnegative, we can shift the degrees by $r+(m-1)$ as in [7, Corollary 4.5.5]. Then the Castelnuovo–Mumford regularity of $\mathrm{Coker}(\bar{\delta}_0)$ is equal to $2r+m-n-3$.

11. Jet modules

In this section, we prove that $\mathrm{Coker}(\bar{\delta}_0) = S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A}) + Q\,S^{\binom{n+m-1}{m-1}})$ in Section 10 is the transpose of the m-jet module $\Omega^{[1,m]}(S/SQ)$. For the basics of jet modules, see [2,3,10].

Let $I := \langle f_1, \ldots, f_k \rangle$ be an ideal of S. Let $R := S/I$. Define jet modules

$$\begin{aligned} &\Omega^{[1,m]}(S) := J_S/J_S^{m+1}, \qquad \Omega^{\leqslant m}(S) := S \otimes_K S/J_S^{m+1}, \\ &\Omega^{[1,m]}(R) := J_R/J_R^{m+1}, \qquad \Omega^{\leqslant m}(R) := R \otimes_K R/J_R^{m+1}, \end{aligned} \tag{11.1}$$

where

$$\begin{aligned} J_S &:= \langle 1 \otimes a - a \otimes 1 \mid a \in S \rangle \subseteq S \otimes_K S, \\ J_R &:= \langle 1 \otimes a - a \otimes 1 \mid a \in R \rangle \subseteq R \otimes_K R. \end{aligned}$$

Then $\Omega^{\leqslant m}(R)$ is the representative object of the functor $M \to D_R^m(R, M)$, i.e., there exists a natural isomorphism of R-modules:

$$D_R^m(R, M) \simeq \mathrm{Hom}_R(\Omega^{\leqslant m}(R), M),$$

where M is an R-module, and $D_R^m(R, M)$ is the module of differential operators of order $\leqslant m$ from R to M.

As S-modules,

$$\Omega^{\leqslant m}(S) = \Omega^{[1,m]}(S) \bigoplus S \otimes 1, \qquad \Omega^{\leqslant m}(R) = \Omega^{[1,m]}(R) \bigoplus R \otimes 1.$$

Here note that S acts as $S \otimes 1$. We have

$$\left\{ P \in D_R^m(R, M) \;\middle|\; P * 1 = 0 \right\} \simeq \mathrm{Hom}_R\left(\Omega^{[1,m]}(R), M\right)$$

for an R-module M.

For $a \in S$ (or R), we denote $1 \otimes a - a \otimes 1$ mod J_S^{m+1} (or J_R^{m+1}, respectively) by da.

Then, for $f, g \in R$, we have

$$d(fg) = f\,dg + g\,df + (df)(dg). \tag{11.2}$$

As an S-module

$$\Omega^{[1,m]}(S) = \bigoplus_{1 \leqslant |\boldsymbol{\alpha}| \leqslant m} S(dx)^{\boldsymbol{\alpha}}.$$

For $f \in S$, we have

$$df = \sum_{1 \leqslant |\boldsymbol{\alpha}| \leqslant m} \frac{1}{\boldsymbol{\alpha}!}\left(\partial^{\boldsymbol{\alpha}} * f\right)(dx)^{\boldsymbol{\alpha}}. \tag{11.3}$$

We have a surjective $S \otimes S$-module homomorphism

$$\varphi : \Omega^{[1,m]}(S) \ni (dx)^{\boldsymbol{\alpha}} \mapsto (d\bar{x})^{\boldsymbol{\alpha}} \in \Omega^{[1,m]}(R).$$

Lemma 11.1. *As an S-module,*

$$\operatorname{Ker} \varphi = \sum_{i;\, 1 \leqslant |\boldsymbol{\alpha}| \leqslant m} S f_i (dx)^{\boldsymbol{\alpha}} + \sum_{i;\, 0 \leqslant |\boldsymbol{\alpha}| \leqslant m-1} S(df_i)(dx)^{\boldsymbol{\alpha}}.$$

Proof. The inclusion '$\supset$' is clear. We prove the other inclusion.
First we prove that

$$\operatorname{Ker} \varphi = I\, dS + S\, dI. \tag{11.4}$$

Clearly the kernel of the $S \otimes S$-module homomorphism:

$$\Omega^{\leqslant m}(S) \ni f \otimes g \mapsto \bar{f} \otimes \bar{g} \in \Omega^{\leqslant m}(R)$$

equals $(S \otimes I + I \otimes S)/J_S^{m+1}$ or $(S\, dI + I \otimes S)/J_S^{m+1}$. Hence, to prove (11.4), it is enough to show that

$$(I \otimes S) \cap J_S = I\, dS. \tag{11.5}$$

Let $\sum_k i_k \otimes g_k \in J_S$ with $i_k \in I$, $g_k \in S$. Then $\sum_k i_k g_k = 0$. We have

$$\sum_k i_k \otimes g_k = \sum_k (i_k \otimes g_k - i_k g_k \otimes 1) + \sum_k i_k g_k \otimes 1 = \sum_k i_k\, dg_k + 0 \in I\, dS.$$

Hence we have proved (11.5) and in turn (11.4). Thus as an S-module

$$\operatorname{Ker} \varphi = \sum_{1 \leqslant |\alpha| \leqslant m} I(dx)^{\boldsymbol{\alpha}} + \sum_{0 \leqslant |\boldsymbol{\alpha}| < m} S dI (dx)^{\boldsymbol{\alpha}}.$$

To finish the proof, we only need to show that $d(f_i x^{\boldsymbol{\alpha}})$ belongs to the right hand of the assertion for any $\boldsymbol{\alpha}$. This is done by (11.2):

$$d\big(f_i x^{\boldsymbol{\alpha}}\big) = f_i d\big(x^{\boldsymbol{\alpha}}\big) + x^{\boldsymbol{\alpha}}\, df_i + (df_i)\big(d\big(x^{\boldsymbol{\alpha}}\big)\big). \qquad \square$$

Hence we have an S-free presentation of $\Omega^{[1,m]}(R)$:

$$\Bigg(\bigoplus_{i;\, 1 \leqslant |\boldsymbol{\alpha}| \leqslant m} S f_i (dx)^{\boldsymbol{\alpha}}\Bigg) \oplus \Bigg(\bigoplus_{i;\, 0 \leqslant |\boldsymbol{\beta}| \leqslant m-1} S(df_i)(dx)^{\boldsymbol{\beta}}\Bigg) \to \Omega^{[1,m]}(S) \to \Omega^{[1,m]}(R) \to 0. \tag{11.6}$$

Now we consider the case $I = SQ$:

$$\Bigg(\bigoplus_{1 \leqslant |\boldsymbol{\alpha}| \leqslant m} SQ\, (dx)^{\boldsymbol{\alpha}}\Bigg) \oplus \Bigg(\bigoplus_{0 \leqslant |\boldsymbol{\beta}| \leqslant m-1} S(dQ)(dx)^{\boldsymbol{\beta}}\Bigg) \to \Omega^{[1,m]}(S) \to \Omega^{[1,m]}(S/SQ) \to 0. \tag{11.7}$$

Hence, as an S/SQ-module, $\Omega^{[1,m]}(S/SQ)$ has a presentation:

$$\bigoplus_{0\leqslant|\boldsymbol{\beta}|\leqslant m-1}(S/SQ)(dQ)(dx)^{\boldsymbol{\beta}}\to\bigoplus_{1\leqslant|\boldsymbol{\alpha}|\leqslant m}(S/SQ)(dx)^{\boldsymbol{\alpha}}\to\Omega^{[1,m]}(S/SQ)\to 0. \tag{11.8}$$

Note that by (11.3)

$$\begin{aligned}(dQ)(dx)^{\boldsymbol{\beta}}&=\sum_{|\boldsymbol{\alpha}+\boldsymbol{\beta}|\leqslant m,\,\boldsymbol{\alpha}\neq\mathbf{0}}\frac{1}{\boldsymbol{\alpha}!}\big(\partial^{\boldsymbol{\alpha}}*Q\big)(dx)^{\boldsymbol{\alpha}+\boldsymbol{\beta}}\\&=\sum_{|\boldsymbol{\gamma}|\leqslant m,\,\boldsymbol{\gamma}\neq\boldsymbol{\beta}}\frac{1}{(\boldsymbol{\gamma}-\boldsymbol{\beta})!}\big(\partial^{\boldsymbol{\gamma}-\boldsymbol{\beta}}*Q\big)(dx)^{\boldsymbol{\gamma}}.\end{aligned}$$

Hence the $(\boldsymbol{\beta},\boldsymbol{\gamma})$-component of the matrix of (11.8) equals $\frac{1}{(\boldsymbol{\gamma}-\boldsymbol{\beta})!}(\partial^{\boldsymbol{\gamma}-\boldsymbol{\beta}}*Q)$.

By Lemma 10.4, the S/SQ-module $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A})+QS^{\binom{n+m-1}{m-1}})$ has a presentation:

$$\begin{aligned}\bigoplus_{1\leqslant|\boldsymbol{\gamma}|\leqslant m}(S/SQ)\frac{1}{\boldsymbol{\gamma}!}\partial^{\boldsymbol{\gamma}}\xrightarrow{\bullet}\bigoplus_{0\leqslant|\boldsymbol{\beta}|\leqslant m-1}&(S/SQ)\boldsymbol{e}_{\boldsymbol{\beta}}\\&\to S^{\binom{n+m-1}{m-1}}/\big(J_m(\mathcal{A})+QS^{\binom{n+m-1}{m-1}}\big)\to 0,\end{aligned}$$

and the $(\boldsymbol{\gamma},\boldsymbol{\beta})$-component of the matrix of the map $\bullet$ in (11.9) (recall (10.1)) equals $\frac{1}{(\boldsymbol{\gamma}-\boldsymbol{\beta})!}(\partial^{\boldsymbol{\gamma}-\boldsymbol{\beta}}*Q)$.

Thus we have proved the following theorem.

Theorem 11.2. *The S/SQ-module $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A})+QS^{\binom{n+m-1}{m-1}})$ is the transpose of $\Omega^{[1,m]}(S/SQ)$.*

Corollary 11.3. *The S/SQ-modules $S^{\binom{n+m-1}{m-1}}/(J_m(\mathcal{A})+QS^{\binom{n+m-1}{m-1}})$ and $\Omega^{[1,m]}(S/SQ)$ share the same Fitting ideals.*

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